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# A note on homotopy types of connected components of Map $(S^4, BSU(2))$

Mitsunobu Tsutaya [1]

*Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan*

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ABSTRACT

Gottlieb has shown that connected components of Map $(S^4, BSU(2))$ are the classifying spaces of gauge groups of principal $SU(2)$-bundles over $S^4$. Tsukuda has investigated the homotopy types of connected components of Map $(S^4, BSU(2))$. But unfortunately, his proof is not complete for $p = 2$. In this paper, we give a complete proof. Moreover, we investigate the further divisibility of $\epsilon_i$ defined in Tsukuda's paper. We apply this to classification problem of gauge groups as $A_n$-spaces.



## 1. Introduction

For a principal $G$-bundle $P$, the gauge group $\mathcal{G}(P)$ of $P$ is defined as the group of all bundle maps $P \to P$ covering the identity map of the base space. It is topologized by compact open topology and becomes a topological group. Particularly, we consider the gauge groups of principal $SU(2)$-bundles over the 4-dimensional sphere $S^4$.

Gottlieb [2] has constructed the universal bundles and the classifying spaces of gauge groups. The classifying space of the gauge group of a principal $G$-bundle $P$ over $B$ with classifying map $\alpha : B \to BG$ is the connected component of the mapping space Map $(B, BG)$ containing $\alpha$.

Kono [4] has investigated the homotopy types of the gauge groups of principal $SU(2)$-bundles over $S^4$. He showed that they are completely classified by the invariant GCD $(12, \langle c_2(P), [S^4]\rangle)$, where GCD $(a, b)$ represents the greatest common divisor of $a$ and $b$ (GCD $(0, a) =$ GCD $(a, 0) = |a|$), $c_2(P) \in H^4(S^4; \boldsymbol{Z})$ is the second Chern class of $P$ and $[S^4] \in H_4(S^4, \boldsymbol{Z})$ is the fundamental class of $S^4$.

Moreover, Crabb and Sutherland [1] classified the $H$-types (equivalent types as $H$-spaces (or Hopf spaces)) of them; they are completely classified by the invariant GCD $(180, \langle c_2(P), [S^4]\rangle)$. Of course, if $\mathcal{G}(P)$ and $\mathcal{G}(P')$ are $H$-equivalent, then they are homotopy equivalent.

Tsukuda [9] has classified the isomorphism classes of them; they are completely classified by $|\langle c_2(P), [S^4]\rangle|$. Moreover, he shows that $\mathcal{G}(P)$ and $\mathcal{G}(P')$ are isomorphic if the classifying spaces of them are homotopy equivalent. We remark that the converse is true in general. But the argument in the proof of Lemma 2.4 of [9] is not valid for $p = 2$. The first aim of the present paper is to give a correct proof of this.

The author [10] has considered the classification problem of the gauge groups by using Stasheff's "$A_n$" [7,8]. Since "$H$-equivalence" and "homotopy equivalence of classifying spaces" are nothing but "$A_2$-equivalence" and "$A_\infty$-equivalence" respectively, this problem is a natural generalization. The second aim is to apply our main theorem to it.

In Section 2, we review the definition of $\epsilon_i$ and the motivation in homotopy theory. In Sections 3 and 4, we investigate the divisibility of $\epsilon_i$. These sections are purely algebraic. In Section 5, we apply our main result to $A_n$-types of gauge groups. We give a lower bound of the number of $A_n$-types of gauge groups of principal $SU(2)$-bundles over $S^4$.

*E-mail address:* tsutaya@math.kyoto-u.ac.jp.

[1] The author is a Research Fellow of the Japan Society for the Promotion of Science.

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## 2. Definition and motivation

We review the definition of $\{\epsilon_i\}$. Let $P_k$ be a principal $SU(2)$-bundle over $S^4$ with $\langle c_2(P_k), [S^4]\rangle = k \in \boldsymbol{Z}$. According to [10], the $p$-localization $\mathcal{G}^{\mathrm{id}}(P_k)_{(p)}$ of the identity component of the gauge group $\mathcal{G}(P_k)$ is $A_n$-equivalent to $\mathcal{G}^{\mathrm{id}}(P_0)_{(p)}$ if and only if the map

$$S^4 \vee \boldsymbol{H}P^n \xrightarrow{k\vee i} \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \xrightarrow{\nabla} \boldsymbol{H}P^\infty \xrightarrow{\text{localization}} \boldsymbol{H}P^\infty_{(p)}$$

extends over $S^4 \times \boldsymbol{H}P^n$, where $k : S^4 \to \boldsymbol{H}P^\infty$ is the classifying map of $P_k$, $i : \boldsymbol{H}P^n \to \boldsymbol{H}P^\infty$ is the inclusion and $\nabla : \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \to \boldsymbol{H}P^\infty$ is the folding map.

Now, we assume there exists the following homotopy commutative diagram:

$$\begin{tikzcd}
S^4 \vee \boldsymbol{H}P^n \arrow[r, "k\vee i"] \arrow[d, "j"'] & \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \arrow[r, "\nabla"] & \boldsymbol{H}P^\infty \arrow[d, "\text{localization}"] \\
S^4 \times \boldsymbol{H}P^n \arrow[rr, "f"] & & \boldsymbol{H}P^\infty_{(p)}
\end{tikzcd}$$

where $p$ is a prime and $j : S^4 \vee \boldsymbol{H}P^n \to S^4 \times \boldsymbol{H}P^n$ is the inclusion.

We denote the localization of the ring of integers by the prime ideal $(p) \subset \boldsymbol{Z}$ by $\boldsymbol{Z}_{(p)}$. The (0th) $p$-local complex $K$-theory $K_{(p)}(\boldsymbol{H}P^\infty_{(p)})$ of $\boldsymbol{H}P^\infty_{(p)}$ is computed as

$$K_{(p)}(\boldsymbol{H}P^\infty_{(p)}) = \boldsymbol{Z}_{(p)}[a].$$

We may assume that the generator $b \in H^4(\boldsymbol{H}P^\infty_{(p)}; \boldsymbol{Q})$ satisfies the equality

$$ch\, a = \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!}.$$

Similarly, take generators $u \in \tilde{K}_{(p)}(S^4)$ and $s \in H^4(S^4; \boldsymbol{Q})$ such that $ch\, u = s$. Then, $f^*b = ks \times 1 + 1 \times b$ in $H^4(S^4 \times \boldsymbol{H}P^n; \boldsymbol{Q})$ and

$$f^*a = ku \times 1 + 1 \times a + \sum_{i=1}^{n} \epsilon_i(k) u \times a^i$$

in $\tilde{K}_{(p)}(S^4 \times \boldsymbol{H}P^n)$, where $\epsilon_i(k) \in \boldsymbol{Z}_{(p)}$. We calculate $f^*ch\, a$ and $ch f^*a$ as follows:

$$f^*ch\, a = f^*\sum_{j=1}^{\infty}\frac{2b^j}{(2j)!} = \sum_{j=1}^{\infty}\frac{2}{(2j)!}(ks\times 1 + 1\times b)^j = ks\times 1 + \sum_{j=1}^{n}\left(\frac{k}{(2j+1)!}s\times b^j + \frac{2}{(2j)!}1\times b^j\right),$$

$$\begin{aligned}
ch f^*a &= ch\left(ku\times 1 + 1\times a + \sum_{i=1}^{\infty}\epsilon_i(k)u\times a^i\right) = ks\times 1 + 1\times\sum_{j=1}^{n}\frac{2}{(2j)!}b^j + \sum_{i=1}^{n}\sum_{j=1}^{n}\epsilon_i(k)s\times\left(\sum_{j=1}^{n}\frac{2}{(2j)!}b^j\right)^i \\
&= ks\times 1 + \sum_{j=1}^{n}\frac{2}{(2j)!}1\times b^j + \sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j_1+\cdots+j_i=l}\frac{2^i\epsilon_i(k)}{(2j_1)!\cdots(2j_i)!}s\times b^l.
\end{aligned}$$

Since $f^*ch\, a = ch f^*a$, we have the following formula:

$$\frac{k}{(2\ell+1)!} = \sum_{i=1}^{\ell}\sum_{\substack{j_1+\cdots+j_i=\ell\\ j_1,\dots,j_i\geq 1}}\frac{2^i\epsilon_i(k)}{(2j_1)!\cdots(2j_i)!}.$$

For $\ell = 1$ and $k \neq 0$, we obtain $\epsilon_1(k)/k = 1/6$. Moreover, using this formula and induction on $i$, one can see that $\epsilon_i(k)/k \in \boldsymbol{Q}$ is independent of $k \neq 0$. Let $\epsilon_i = \epsilon_i(k)/k$. Then $\epsilon_i(k) = \epsilon_i k$ for any $k \in \boldsymbol{Z}$. Of course, the sequence $\{\epsilon_i\}_{i=1}^{\infty}$ satisfy the following formula for each $\ell$:

$$\frac{1}{(2\ell+1)!} = \sum_{i=1}^{\ell}\sum_{\substack{j_1+\cdots+j_i=\ell\\ j_1,\dots,j_i\geq 1}}\frac{2^i\epsilon_i}{(2j_1)!\cdots(2j_i)!}.$$

For example, $\epsilon_1 = 1/6$, $\epsilon_2 = -1/180$, $\epsilon_3 = 1/1512$ etc. From the above argument, if the map $(\text{localization})\nabla(k\vee i) : S^4 \vee \boldsymbol{H}P^n \to \boldsymbol{H}P^\infty_{(p)}$ extends over $S^4 \times \boldsymbol{H}P^n$, then $\epsilon_1 k, \dots, \epsilon_n k \in \boldsymbol{Z}_{(p)}$.

Tsukuda [9] defines a non-negative integer (or infinity) $d_p(k)$ for a prime $p$ and an integer $k$ as the largest $n$ such that there exists an extension of

$$S^4 \vee \boldsymbol{H}P^n \xrightarrow{k\vee i} \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \xrightarrow{\nabla} \boldsymbol{H}P^\infty \xrightarrow{\text{localization}} \boldsymbol{H}P^\infty_{(p)}$$

over $S^4 \times \boldsymbol{H}P^n$. Remark $d_p(0) = \infty$. Clearly, if we define $\epsilon_0 = 1$, then

$$d_p(k) \le d'_p(k) := \min\{\, n \in \boldsymbol{Z}_{\ge 0} \mid \epsilon_{n+1} k \notin \boldsymbol{Z}_{(p)} \,\}.$$

It is shown in Lemma 2.5 of [10] that $d_p(k) = d_p(k')$ for any prime $p$ if the classifying spaces $B\mathcal{G}(P_k)$ and $B\mathcal{G}(P_{k'})$ are homotopy equivalent. Lemma 2.4 in [9] asserts that $d'_p(k) < \infty$ (therefore $d_p(k) < \infty$) for $k \ne 0$ and any prime $p$. In the proof of it, he has shown that $k/(2n+1) \in \boldsymbol{Z}_{(p)}$ if $n \le d'_p(k)$. For an odd prime $p$, if $p^r \not| \; k$, this implies that $d'_p(k) < (p^r - 1)/2$. But for $p = 2$, this gives no information about $d'_2(k)$. So this proof is invalid for $p = 2$. We will give a correct proof for this case in Section 4.

We also state the result of [10]. If $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are $A_n$-equivalent, then $\min\{n, d_p(k)\} = \min\{n, d_p(k')\}$ for any prime $p$. Using this, we give a lower bound for the number of $A_n$-types of the gauge groups.

## 3. An explicit formula for $\epsilon_i$

As we have seen, algebraically, the sequence $\{\epsilon_i\}_{i=0}^{\infty}$ of rational numbers is defined inductively by the following formula:

$$\frac{1}{(2\ell+1)!} = \sum_{i=1}^{\ell} \sum_{\substack{j_1+\cdots+j_i=\ell \\ j_1,\ldots,j_i \ge 1}} \frac{2^i \epsilon_i}{(2j_1)!\cdots(2j_i)!}$$

and $\epsilon_0 = 1$. Equivalently, $\{\epsilon_i\}$ is defined by the equality

$$\sum_{\ell=0}^{\infty} \frac{x^\ell}{(2\ell+1)!} = \sum_{i=0}^{\infty} \epsilon_i \left( \sum_{j=1}^{\infty} \frac{2x^j}{(2j)!} \right)^i$$

in the ring of formal power series $\boldsymbol{Q}[[x]]$.

**Proposition 3.1.** *The rational number $\epsilon_i$ is the i-th coefficient of the Taylor expansion of $1/F'(x)$ at $0 \in \boldsymbol{C}$ for*

$$F(x) = \left( \cosh^{-1}\left(1 + \frac{x}{2}\right)\right)^2,$$

*where $F$ is holomorphic in a neighbourhood of* 0.

**Proof.** Define a holomorphic function $H$ by

$$H(x) = 2\cosh\sqrt{x} - 2 = \sum_{i=1}^{\infty} \frac{2}{(2i)!} x^i$$

in a neighbourhood of 0. Then $F$ given by the above formula is the inverse function of $H$. We also define $G$ by

$$G(x) = \sum_{i=0}^{\infty} \epsilon_i x^i.$$

Then, formally, $H'(x) = G(H(x))$ by the definition of $\epsilon_i$. Therefore, we have $G(x) = 1/F'(x)$. □

Since

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2+1}},$$

the next proposition is seen by easy computation.

**Proposition 3.2.** *The holomorphic function $F$ satisfies the following differential equation:*

$$x(x+4)F''(x) + (x+2)F'(x) - 2 = 0.$$

If the power series

$$\sum_{i=1}^{\infty} a_i x^i$$

satisfies the above equation, then

$$a_1 = 1,\; a_{i+1} = -\frac{i^2}{(2i+2)(2i+1)} a_i \quad (i \ge 1).$$

From these equations,

$$a_i = (-1)^{i-1} \frac{2((i-1)!)^2}{(2i)!}$$

for $i \geq 1$. Hence,

$$F(x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2((i-1)!)^2}{(2i)!} x^i$$

and

$$F'(x) = \sum_{i=0}^{\infty} (-1)^i \frac{(i!)^2}{(2i+1)!} x^i.$$

Therefore,

$$G(x) = \frac{1}{F'(x)} = \sum_{j=0}^{\infty} (-1)^j \left( \sum_{i=1}^{\infty} (-1)^i \frac{(i!)^2}{(2i+1)!} x^i \right)^j = 1 + \sum_{j=1}^{\infty} \sum_{i_1,\dots,i_j \geq 1} (-1)^{j+i_1+\cdots+i_j} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!} x^{i_1+\cdots+i_j}.$$

This implies the following formula.

**Theorem 3.3.**

$$\epsilon_\ell = \sum_{j=1}^{\ell} \sum_{\substack{i_1+\cdots+i_j=\ell \\ i_1,\dots,i_j \geq 1}} (-1)^{j+\ell} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}$$

## 4. Divisibility of $\epsilon_i$

For a prime $p$ and a rational number $n$, we denote the $p$-adic valuation of $n$ by $v_p(n)$. Equivalently, if

$$n = \frac{p^a t}{p^b s}$$

where $s$ and $t$ are integers prime to $p$, then $v_p(n) = a - b$. First, we review the divisibility of factorials. For a positive integer $n$ and each prime $p$, the following formula is known (see, for example, [5]):

$$v_p(n!) = \frac{n - S_p(n)}{p - 1},$$

where $S_p(n) = n_0 + \cdots + n_r$ for $n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_0$ and $0 \leq n_i < p$.

**Lemma 4.1.** *For a positive integer $n$*

$$v_2\left( \frac{(n!)^2}{(2n+1)!} \right) \geq -n,$$

*where the equality holds if and only if $n = 1$.*

**Proof.** Using the above formula, we obtain

$$v_2\left( \frac{(n!)^2}{(2n+1)!} \right) = 2(n - S_2(n)) - (2n + 1 - S_2(2n+1)) = S_2(n) \geq -n.$$

As easily seen, this equality holds if and only if $n = 1$. □

Now, we observe the divisibility of $\epsilon_i$ by 2.

**Proposition 4.2.**

$$v_2(\epsilon_\ell) = -\ell$$

**Proof.** By Theorem 3.3,

$$\epsilon_\ell = \sum_{j=1}^{\ell} \sum_{\substack{i_1+\cdots+i_j=\ell \\ i_1,\dots,i_j \geq 1}} (-1)^{j+\ell} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}.$$

Now, from the previous lemma, we have

$$v_2\left( \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!} \right) \geq -i_1 - \cdots - i_j = -\ell,$$

where the equality holds if and only if $i_1 = \cdots = i_j = 1$. Then we obtain

$$\epsilon_\ell \equiv 6^{-\ell} \mod 2^{-\ell+1} \mathbf{Z}_{(2)}.$$

Therefore, $v_2(\epsilon_\ell) = -\ell$. □

In general, the divisibility of $\epsilon_i$ by an odd prime $p$ is more complicated than by 2 because the interval between a multiple of $p$ and the next one is longer. But for $p = 3$, we will have a similar result.

**Lemma 4.3.** *Let $p$ be an odd prime and $n$ a positive integer. Then*

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \geq -\frac{2n}{p-1},$$

*where the equality holds if and only if $n = (p-1)/2$.*

**Proof.** Let $n = n_r p^r + n_{r-1}p^{r-1} + \cdots + n_0$ with $0 \leq n_i < p$. Assume $n_0 = \cdots = n_{s-1} = (p-1)/2 \neq n_s$ $(0 \leq s \leq r+1)$. Let $m = n_r p^{r-s} + \cdots + n_s$ for $s \leq r$ and $m = 0$ for $s = r+1$. Then we have $2n+1 = p^s(2m+1)$, where $2m+1$ is prime to $p$ since $n_s \neq (p-1)/2$. Thus we obtain

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) = v_p\left(\frac{1}{2n+1}\right) + v_p\left(\frac{(n!)^2}{(2n)!}\right) = -s - \frac{2S_p(n) - S_p(2n)}{p-1} = -s - \frac{2S_p(m) - S_p(2m)}{p-1} \geq -s - \frac{2m}{p-1},$$

where the last equality holds if and only if $m = 0$. Moreover,

$$\frac{2(n-m)}{p-1} = \frac{(2n+1)-(2m+1)}{p-1} = \frac{(p^s-1)(2m+1)}{p-1} \geq 1 + p + \cdots + p^{s-1} \geq s,$$

where $2(n-m)/(p-1) = s$ if and only if $s = 1$ and $m = 0$. Therefore, we conclude that

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \geq -\frac{2n}{p-1},$$

where the equality holds if and only if $n = (p-1)/2$. $\square$

**Proposition 4.4.** *For a positive integer $\ell \leq n(p-1)/2$,*

$$v_p(\epsilon_\ell) \geq -n,$$

*where the equality holds if and only if $\ell = n(p-1)/2$. Especially, $v_3(\epsilon_\ell) = -\ell$ for any $\ell$.*

**Proof.** Let positive integers $i_1, \ldots, i_j$ satisfy $i_1 + \cdots + i_j = \ell$. From the previous proposition, we have

$$v_p\left(\frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}\right) \geq -\frac{2\ell}{p-1} \geq -n,$$

where the left equality holds if and only if $i_1 = \cdots = i_j = (p-1)/2$. Thus using Theorem 3.3, we obtain

$$v_p(\epsilon_\ell) \geq -n,$$

where the equality holds if and only if $\ell = n(p-1)/2$. $\square$

**Remark 1.** Here we remark that $v_p(\epsilon_\ell)$ is not non-increasing on $\ell$ in general. For example, $v_5(\epsilon_2) = -1$ but $v_5(\epsilon_3) = 0$, $v_7(\epsilon_9) = -3$ but $v_7(\epsilon_{10}) = -2$, etc.

From Propositions 4.2 and 4.4, we conclude our main result.

**Theorem 4.5.** *Let $k$ be a non-zero integer and $p$ be a prime. Then*

$$d'_p(k) = \begin{cases} v_2(k) & (p = 2) \\ v_p(k)(p-1)/2 & (p \text{ is odd}). \end{cases}$$

Then $d_2(k) \leq d'_2(k) < \infty$ for any non-zero integer $k$ and we have correctly verified Lemma 2.4 of [9].

## 5. Applications to $A_n$-types of gauge groups

As in Section 2, we assume there exists the following homotopy commutative diagram:

$$\begin{tikzcd} S^4 \vee \boldsymbol{H}P^n \arrow[r, "k \vee i"] \arrow[d, "j"] & \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \arrow[r, "\nabla"] & \boldsymbol{H}P^\infty \arrow[d, "\text{localization}"] \\ S^4 \times \boldsymbol{H}P^n \arrow[rr, "f"] & & \boldsymbol{H}P^\infty_{(p)} \end{tikzcd}$$

where $p$ is a prime and $i$ and $j$ are the inclusions. Let us consider the map

$$S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \overset{f\cup((\text{localization})i)}{\to} \boldsymbol{H}P^\infty_{(p)}.$$

The obstruction to extending this map over $S^4 \times \boldsymbol{H}P^{n+1}$ lies in $\pi_{4n+7}(\boldsymbol{H}P^\infty_{(p)})$. Then, from Theorem of [6], the obstruction to extending the map

$$S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \overset{(p\times \mathrm{id})\cup \mathrm{id}}{\to} S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \overset{f\cup((\text{localization})i)}{\to} \boldsymbol{H}P^\infty_{(p)}$$

over $S^4 \times \boldsymbol{H}P^{n+1}$ vanishes for an odd prime $p$. Here we remark that for the cofibration

$$S^{4n+7} \overset{\varphi}{\to} S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \to S^4 \times \boldsymbol{H}P^{n+1},$$

$\varphi : S^{4n+7} \to S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1}$ is defined by the restriction to the boundary of the direct product of the characteristic maps $(D^4, S^3) \to (S^4, *)$ and $(D^{4n+4}, S^{4n+3}) \to (\boldsymbol{H}P^{n+1}, \boldsymbol{H}P^n)$ and the composition of the direct product and the restriction $\pi_4(S^4) \times \pi_{4n+4}(\boldsymbol{H}P^{n+1}, \boldsymbol{H}P^n) \to \pi_{4n+8}(S^4 \times \boldsymbol{H}P^{n+1}, S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1}) \to \pi_{4n+7}(S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1})$ is bilinear. Hence one can see $d_p(pk) > d_p(k)$ and $d_p(k) \geq \upsilon_p(k)$ inductively. For $p = 2$, from [3], $d_2(4k) > d_2(k)$ and $d_2(k) \geq [\upsilon_2(k)/2]$ similarly. Then, combining with Theorem 4.5, we have

$$\upsilon_p(k) \leq d_p(k) \leq d'_p(k) = \frac{p-1}{2}\upsilon_p(k)$$

for an odd prime $p$ and

$$\left[\frac{\upsilon_2(k)}{2}\right] \leq d_2(k) \leq d'_2(k) = \upsilon_2(k).$$

Especially, $d_3(k) = \upsilon_3(k)$. From Section 2, the next theorem follows.

**Theorem 5.1.** *Let $P$ be a principal $SU(2)$-bundle over $S^4$ with $\langle c_2(P), [S^4]\rangle = 3^n$. Then the 3-localization $\mathcal{G}^{\mathrm{id}}(P)_{(3)}$ of the identity component $\mathcal{G}^{\mathrm{id}}(P)$ of $\mathcal{G}(P)$ is $A_n$-equivalent to* Map $(S^4, SU(2))_{(3)}$ *but not $A_{n+1}$-equivalent.*

This gives an example of $A_n$-equivalent but not $A_{n+1}$-equivalent topological monoids for any $n$.

Now we give the lower bound of the number of $A_n$-types of gauge groups of principal $SU(2)$-bundle over $S^4$. As stated in Section 2, if $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are $A_n$-equivalent, then $\min\{n, d_p(k)\} = \min\{n, d_p(k')\}$ for any prime $p$. If $p$ is an odd prime, then

$$\#\{\min\{n, d_p(k)\} \mid k \in \boldsymbol{Z}\} \geq \left[\frac{2n}{p-1} + 1\right]$$

since $0 = d_p(1) < d_p(p) < d_p(p^2) < \cdots < d_p(p^{[2n/(p-1)]}) \leq n$. If $p = 2$, then

$$\#\{\min\{n, d_2(k)\} \mid k \in \boldsymbol{Z}\} \geq \left[\frac{n}{2} + 1\right]$$

since $0 = d_2(1) < d_2(4) < d_2(16) < \cdots < d_2(4^{[n/2]}) \leq n$.

**Theorem 5.2.** *The number of $A_n$-types of gauge groups of principal $SU(2)$-bundles over $S^4$ is greater than*

$$\left[\frac{n}{2} + 1\right] \prod_{p:\text{odd prime}} \left[\frac{2n}{p-1} + 1\right].$$

We can express the logarithm of this as follows:

$$\begin{aligned}\log\left(\left[\frac{n}{2} + 1\right] \prod_{p:\text{odd prime}} \left[\frac{2n}{p-1} + 1\right]\right) &= \log\left[\frac{n}{2} + 1\right] + \sum_{p:\text{odd prime}} \log\left[\frac{2n}{p-1} + 1\right]\\ &= \log\left[\frac{n}{2} + 1\right] + \sum_{r=2}^{n+1}\left(\pi\left(\frac{2n}{r-1} + 1\right) - 1\right)(\log r - \log(r-1))\\ &= \log\left[\frac{n}{2} + 1\right] + \sum_{r=1}^{n}\pi\left(\frac{2n}{r} + 1\right)\log\left(1 + \frac{1}{r}\right) - \log(n+1),\end{aligned}$$

where $\pi$ is the prime counting function. The second equality is seen by

$$\#\left\{p : \text{an odd prime} \;\middle|\; \left[\frac{2n}{p-1} + 1\right] \geq r\right\} = \pi\left(\frac{2n}{r-1} + 1\right) - 1.$$

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